SCOTCH COLLEGE

Year 12 Specialist Mathematics

TEST 3 – Integration

Part A – Calculator Free

Name: SOLUTIONS **Teacher: Fugill | Langley** ***(please circle)***

Date: 6th April 2016

SECTION ONE: CALCULATOR FREE

TOTAL: 35 marks

EQUIPMENT: Pens, pencils, pencil sharpener, highlighter, eraser, ruler, SCSA formula sheet.

WORKING TIME: 35 minutes

SECTION TWO: CALCULATOR ASSUMED

TOTAL: 19 marks

EQUIPMENT: Pens, pencils, pencil sharpener, highlighter, eraser, ruler, drawing instruments, templates, up to 3 Calculators,

1 A4 page of notes (one side only), SCSA formula sheet.

WORKING TIME: 20 minutes

Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

Question	Marks available	Marks awarded
1	10	
2	8	
3	4	
4	6	
5	7	
6	5	
7	6	
8	8	
Total	**54**	

Calculator-free **[35 marks]**

This paper has **Five (5)** questions. Answer **all** questions. Write your answers in the spaces provided

Question 1 [10 marks]

Determine the following indefinite integrals.

a) $\int \frac{x^2+2x-5}{x^2}\,dx$ **[3]**

$\int 1+2x^{-1}-5x^{-2}dx$ ✓ simplify integral

$= x+2\ln|x|+\frac{5}{x}+c$ ✓✓

b) $\int \sin^3(2x)\cos(2x)\,dx$ **[3]**

$= \frac{1}{2}\int 2\cos(2x)\sin^3(2x)\,dx$

$= \frac{1}{2}\frac{\sin^4(2x)}{4}+c$ ✓ ✓

$= \frac{\sin^4(2x)}{8}+c$ ✓

c) $\int x\sqrt{x-1}\;dx$ **[4]**

$let\ u = x-1 \quad \Rightarrow x = u+1$

$\frac{du}{dx} = 1$ ✓

$\int (u+1)\sqrt{u}\;du$ ✓

$\int u^{\frac{3}{2}}+u^{\frac{1}{2}}\;du$

$\frac{2}{5}u^{\frac{5}{2}}+\frac{2}{3}u^{\frac{3}{2}}+c$ ✓

$\frac{2}{5}(x-1)^{\frac{5}{2}}+\frac{2}{3}(x-1)^{\frac{3}{2}}+c$ ✓

Question 2 [8 marks]

Evaluate:

a) $\int_0^{\frac{\pi}{4}} \cos 2x \, dx$ **[3]**

$$= \left[\frac{\sin(2x)}{2}\right]_0^{\frac{\pi}{4}} \checkmark$$

$$= \frac{\sin\left(\frac{\pi}{2}\right)}{2} - \frac{\sin(0)}{2} \checkmark$$

$$= \frac{1}{2} \checkmark$$

b) $\int_0^1 \frac{x-1}{x^2+3x+2} \, dx$ **[5]**

$$\frac{x-1}{x^2+3x+2} = \frac{x-1}{(x+2)(x+1)}$$

$$\frac{x-1}{(x+2)(x+1)} =$$

$$x-1 = A(x+1) + B(x+2)$$

$$let\ x = -1 \Rightarrow -2 = B \checkmark$$

$$let\ x = -2 \Rightarrow -3 = -A \Rightarrow A = 3$$

$$\int_0^1 \frac{3}{x+2} - \frac{2}{x+1} dx \checkmark$$

$$= \left[3\ln|x+2| - 2\ln|x+1|\right]_0^1 \checkmark$$

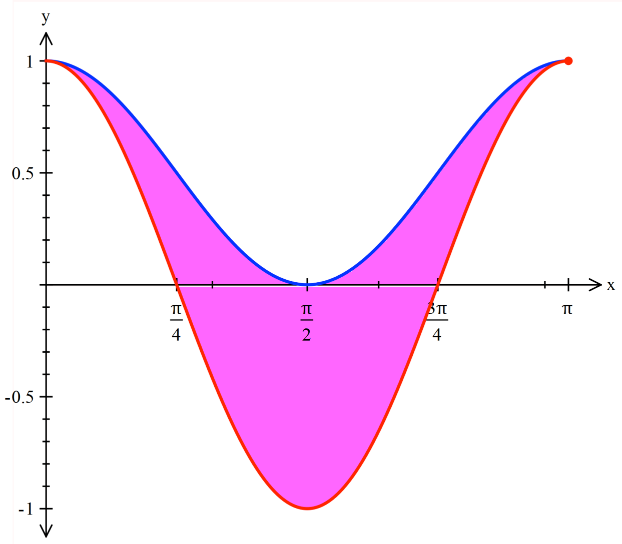
$$= \left(3\ln|3| - 2\ln|2|\right) - \left(3\ln|2| - 2\ln|1|\right) \checkmark$$

$$= 3\ln|3| - 5\ln|2| \checkmark$$

$$= \ln\frac{27}{32}$$

Question 3 [4 marks]

The curves below are $y = \cos(2x)$ and $y = \cos^2 x$. Determine the area of the shaded region.



$$\int_0^{\pi} \cos^2 x - \cos(2x)\, dx \quad ✓$$

$$\int_0^{\pi} \left(\frac{1+\cos(2x)}{2}\right) - \cos(2x)\, dx$$

$$\int_0^{\pi} \frac{1}{2} - \frac{1}{2}\cos(2x)\, dx \quad ✓$$

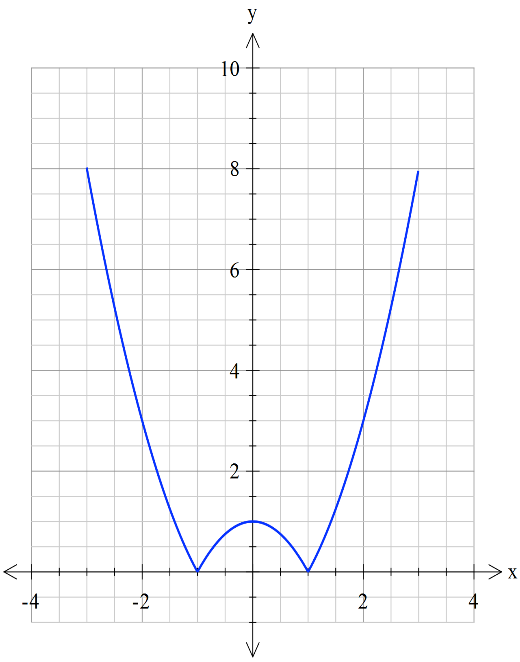
$$\left[\frac{x}{2} - \frac{1}{4}\sin(2x)\right]_0^{\pi} \quad ✓$$

$$\left(\frac{\pi}{2} - \frac{1}{4}\sin(2\pi)\right) - \left(\frac{0}{2} - \frac{1}{4}\sin(0)\right)$$

$$\frac{\pi}{2} units^2 \quad ✓$$

Question 4 [6 marks]

A glass is formed by rotating the function $f(x) = |x^2 - 1|$ about the y-axis as shown in the diagram.



a) Complete the middle line for the piecewise function for $f(x) = |x^2 - 1|$.

$$f(x) = \begin{cases} x^2 - 1 \; ; \; x \le -1 \\ -x^2 + 1 \; ; -1 < x < 1 \\ x^2 - 1 \; ; \; x \ge 1 \end{cases}$$ ✓

[1]

b) Given the height of the glass is 8 cm determine the volume of the glass. **[5]**

✓ ✓

$$\pi \int_0^8 \left(\sqrt{y+1}\right)^2 dy - \pi \int_0^1 \left(\sqrt{1-y}\right)^2 dy$$

$x = \sqrt{y+1}$
and
$x = \sqrt{1-y}$

$$= \pi \int_0^8 y + 1 \, dy - \pi \int_0^1 1 - y \, dy$$

$$= \pi \left[\frac{y^2}{2} + y\right]_0^8 - \pi \left[y - \frac{y^2}{2}\right]_0^1$$ ✓

$$= \pi \left[\left(\frac{8^2}{2} + 8\right) - (0)\right] - \pi \left[\left(1 - \frac{1}{2}\right) - (0)\right]$$ ✓

$$= 40\pi - \frac{1}{2}\pi$$

$$= 39.5\pi cm^3$$ ✓

Question 5 [7 marks]

Find $\int_{1}^{\sqrt{3}} \sqrt{4-x^2}\,dx$ using the substitution $x = 2 \sin \theta$.

$$\frac{dx}{d\theta} = 2\cos\theta \quad \Rightarrow \quad dx = 2\cos\theta\, d\theta \quad ✓$$

change the limits:

$$1 = 2\sin\theta \qquad \sqrt{3} = 2\sin\theta$$

$$\frac{1}{2} = \sin\theta \qquad \frac{\sqrt{3}}{2} = \sin\theta$$

$$\theta = \frac{\pi}{6} \ ✓ \qquad \theta = \frac{\pi}{3} \ ✓$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sqrt{4-4\sin^2\theta}\ \ 2\cos\theta\, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 2\sqrt{1-\sin^2\theta}\ \ 2\cos\theta\, d\theta \quad ✓$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 2\sqrt{\cos^2\theta}\ \ 2\cos\theta\, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 2\cos\theta\ \ 2\cos\theta\, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 4\cos^2\theta\, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 4\left(\frac{1+\cos 2\theta}{2}\right) d\theta \quad ✓$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 2 + 2\cos 2\theta\, d\theta$$

$$= \left[2\theta + \sin 2\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} \quad ✓$$

$$= \left(\frac{2\pi}{3} - \sin\frac{2\pi}{3}\right) - \left(\frac{2\pi}{6} - \sin\frac{2\pi}{6}\right)$$

$$= \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) - \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)$$

$$= \frac{\pi}{3} \quad ✓$$

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TEST 3 – Integration

Part B – Calculator Allowed

Name:______________________________ **Teacher: Fugill | Langley** ***(please circle)***

Calculator Allowed **20 minutes** **[19 marks]**

This paper has **Three (3)** questions. Answer **all** questions. Write your answers in the spaces provided

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Question 6 [5 marks]

Determine the value of p in the system of linear equations below such that there is

$$x - 2y - 3z = 11$$
$$2x - y + z = 5$$
$$3x + 3y + pz = -6$$

a) no solution **[3]**

$$\begin{bmatrix} 1 & -2 & -3 & 11 \\ 2 & -1 & 1 & 5 \\ 3 & 3 & p & -6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & -3 & 11 \\ 0 & 3 & 7 & -17 \\ 0 & 9 & p+9 & -39 \end{bmatrix}$$ ✓ ✓

$$\begin{bmatrix} 1 & -2 & -3 & 11 \\ 0 & 3 & 7 & -17 \\ 0 & 0 & p-12 & 12 \end{bmatrix}$$

$p = 12$ ✓

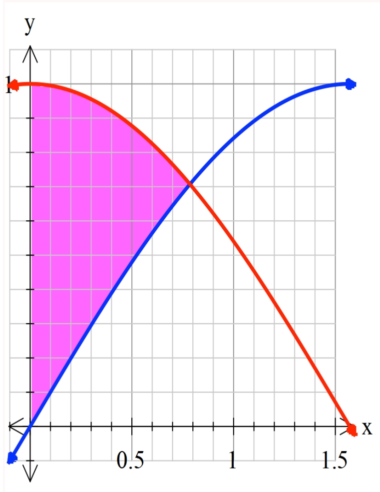
b) a unique solution **[1]**

$p \neq 12$ ✓

c) infinitely many solutions **[1]**

Not possible ✓

Question 7 [6 marks]



The graph at the right show the curves $y = \cos x$ and $y = \sin x$.

a) Prove that the intersection of $y = \cos x$ and $y = \sin x$, is $x = \dfrac{\pi}{4}$ for the domain $0 \le x \le \dfrac{\pi}{2}$.

$\sin x = \cos x$

$\dfrac{\sin x}{\cos x} = 1$

$\tan x = 1$ ✓

$x = \dfrac{\pi}{4}$

[1]

Do not accept showing that $\sin\left(\dfrac{\pi}{4}\right) = \dfrac{1}{\sqrt{2}}$ and $\cos\left(\dfrac{\pi}{4}\right) = \dfrac{1}{\sqrt{2}}$

b) Determine the exact area of the region (shaded) which is bounded by the *y*-axis and the curves $y = \cos x$ and $y = \sin x$. **[2]**

$\int_0^{\frac{\pi}{4}} \cos x - \sin x \, dx$ ✓

$= \left[\sin x + \cos x\right]_0^{\pi/4}$

$= \left(\sin\dfrac{\pi}{4} + \cos\dfrac{\pi}{4}\right) - \left(\sin 0 + \cos 0\right)$

$= \dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2} - 1$

$= \left(\sqrt{2} - 1\right) units^2$ ✓

c) Determine the volume of revolution obtained when this area is rotated about the *x*-axis. **[3]**

$\pi \int_0^{\frac{\pi}{4}} \cos^2 x - \sin^2 x \, dx$ ✓

$= \pi \int_0^{\frac{\pi}{4}} \cos 2x \, dx$

$= \pi \left[\dfrac{\sin 2x}{2}\right]_0^{\pi/4}$ ✓

$= \pi \left[\left(\dfrac{\sin\left(\dfrac{\pi}{2}\right)}{2}\right) - \left(\dfrac{\sin 0}{2}\right)\right]$

$= \dfrac{\pi}{2} units^3$ ✓

Question 8 [8 marks]

a) Use the identity $\cos 2\theta = 2\cos^2\theta - 1$ to prove that $\cos\left(\frac{1}{2}x\right) = \sqrt{\frac{1+\cos x}{2}},\ 0 \le x \le \pi.$ **[2]**

$$\cos x = 2\cos^2\left(\frac{1}{2}x\right) - 1 \quad ✓$$

$$\frac{\cos x + 1}{2} = \cos^2\left(\frac{1}{2}x\right)$$

$$\sqrt{\frac{\cos x + 1}{2}} = \cos\left(\frac{1}{2}x\right) \quad ✓$$

b) Find a similar expression for $\sin\left(\frac{1}{2}x\right),\ 0 \le x \le \pi.$ **[2]**

$$\cos x = 1 - 2\sin^2\left(\frac{1}{2}x\right) \quad ✓$$

$$\sin^2\left(\frac{1}{2}x\right) = \frac{1-\cos x}{2}$$

$$\sin\left(\frac{1}{2}x\right) = \sqrt{\frac{1-\cos x}{2}} \quad ✓$$

c) Hence show that $\int_0^{\frac{\pi}{2}} (\sqrt{1+\cos x} + \sqrt{1-\cos x})\ dx = 2\sqrt{2}$ **[4]**

$$\sqrt{2}\int_0^{\pi/2} \cos\left(\frac{1}{2}x\right) + \sin\left(\frac{1}{2}x\right)\ dx \quad ✓ ✓$$

$$= \sqrt{2}\left[2\sin\left(\frac{1}{2}x\right) - 2\cos\left(\frac{1}{2}x\right)\right]_0^{\pi/2} \quad ✓$$

$$= \sqrt{2}\left[\left(2\sin\left(\frac{\pi}{4}\right) - 2\cos\left(\frac{\pi}{4}\right)\right) - \left(2\sin(0) - 2\cos(0)\right)\right]$$

$$= \sqrt{2}\left(\left(\sqrt{2} - \sqrt{2}\right) - (0-2)\right)$$

$$= 2\sqrt{2} \quad ✓$$